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RECOVERY OF BOUNDARIES BY A VARIATIONAL METHOD I: THE PIECEWISE CONSTANT CASE

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Abstract

A variational method for the reconstruction and segmentation of images was recently proposed by Mumford and Shah [7]. See also [1]. In this paper we study minimizers of one of the limiting versions of the functional introduced in [7]. We show that asymptotically one recovers the discontinuity set of a piecewise constant image. We allow for smearing and additive (bounded) noise.

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1 Introduction

In [7] Mumford and Shah introduced the idea of minimizing an ad hoc energy functional to do image segmentation. In this paper we consider one of the limiting versions of the functional they introduced in which the image g is approximated by a piecewise constant function f, the boundaries in the segmentation being given by the discontinuity set of f. In [8] Mumford and Shah studied the first variation of the functional and found various necessary conditions on the geometry of the boundaries of minimizers. It seems that these necessary conditions indicate little about how well the approach performs in image segmentation. In fact one could argue that some of the conditions constitute negative results in the sense that they place restrictions on the boundaries in the segmentation that do not conform to our expectations of what boundaries of objects should be. These restrictions, however, are of a local nature (being discovered by considering local variations of the boundaries,) and do not necessarily reflect the overall performance of the approach. In this paper we present a result which is of a positive nature. We consider the case in which g is a corrupted version of an otherwise piecewise constant function. We show that in a some limiting sense one can recover the "true" boundaries of the image by the variational approach.

2 Framework of the Problem

2.1 The Variational Problem

Let $\Omega \subset \Re^2$ be an open rectangle. We will be examining minimizers of

$$E(f,\Gamma) = \sum_{k=1}^{\infty} \int_{\Omega_k} (g-f)^2 + \lambda \mathcal{H}^1(\Gamma)$$

where $\lambda > 0$ is a real parameter, the $\Omega_k \in \Omega$ are disjoint open connected sets, $\Gamma = \Omega \backslash \cup_k \Omega_k$, and f is a function constant on each Ω_k. $\mathcal{H}^1$ is the one-dimensional Hausdorff measure which is defined below. It is easy to see that minimality of E requires $f = \frac{1}{|\Omega_k|}\int_{\Omega_k} g$ in Ω_k ($|\cdot|$ denotes lebesgue measure,) so minimizing solutions are determined by Γ and we will often refer to the solution Γ meaning the pair f, Γ. Also, we will be varying the parameter λ and will use Γ_λ to indicate an optimal solution for a particular value of λ. For a given g and λ we define

$$E^*(g, \lambda) = \inf\{E(f, \Gamma)\}$$

The goal of this paper is to show that for g which are approximately piecewise constant the minimizers of E return boundaries which approximate the "true" boundaries of g. The result is asymptotic in nature stating that for λ sufficienctly small and g sufficiently close to a piecewise constant function the Hausdorff distance (defined below) between the "true" boundaries of g and Γ is arbitrarily small. The techniques used here are elementary, admitting constructive analysis useful perhaps for computation and extension to non-asymptotic results. The assumptions are not the most general possible but to weaken them requires the introduction of much more sophisticated techniques into the proof; also, they are certainly general enough for vision applications.

2.2 The Hausdorff Metric

For $A \subset \Re^n$, the *ϵ-neighborhood* of A will be denoted by $[A]_\epsilon$ and is defined by

$$[A]_\epsilon = \{x \in \Re^n : \inf_{y \in A} \|x - y\| < \epsilon\}$$

The notion of distance between boundary sets which we will use is the Hausdorff metric $d_H(\cdot,\cdot)$;

$$d_H(A_1, A_2) = \inf\{\epsilon : A_1 \subset [A_2]_\epsilon \text{ and } A_2 \subset [A_1]_\epsilon\}$$

It is elementary to show that $d_H(\cdot,\cdot)$ is in fact a metric on the space of all non-empty compact subsets of $\Re^n$. The following theorem is a completeness result for subspaces of this metric space.

Theorem 1 *Let C be an infinite collection of non-empty closed subsets of a bounded closed set $\overline{\Omega}$. Then there exists a sequence $\{\Gamma_n\}$ of distinct sets of C and a non-empty closed set $\Gamma \subset \overline{\Omega}$ such that $\Gamma_n \to \Gamma$ in the Hausdorff metric.*

Proof: See [2], Theorem 3.16. ∎

2.3 Hausdorff Measure

The one-dimensional Hausdorff measure is defined as follows. For a non-empty subset A of $\Re^n$, the *diameter* of A is defined by $\mathrm{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$ where $\|\cdot\|$ denotes the Euclidean norm. For $\delta > 0$ set,

$$\mathcal{H}^1_\delta(A) = \inf\{\sum_{i=1}^{\infty} \mathrm{diam}(U_i) \ : A \subset \bigcup_{i=1}^{\infty} U_i,\ \mathrm{diam}(U_i) \leq \delta\}$$

The *Hausdorff 1-dimensional measure* of A is then given by

$$\mathcal{H}^1(A) = \lim_{\delta \to 0} \mathcal{H}^1_\delta(A) = \sup_{\delta > 0} \mathcal{H}^1_\delta(A)$$

Many properties of Hausdorff measure can be found in [2,3,10]. The following definitions are required to state several useful properties. A *curve* $\Gamma \subset \Re^n$ is the image of a continuous injection $g : [0,1] \to \Re^n$. The *length* of a curve Γ is defined as

$$L(\Gamma) = \sup\{\sum_{i=1}^{m} \|g(t_i) - g(t_{i-1})\| \ : \ 0 = t_0 < t_1 < \cdots < t_m = 1\}$$

and Γ is said to be *rectifiable* if $L(\Gamma) < \infty$. Finally, a compact connected set is called a *continuum.*

Theorem 2 *If $\Gamma \subset \Re^n$ is a curve, then $\mathcal{H}^1(\Gamma) = L(\Gamma)$.*

Proof: See [2] Lemma 3.2. ∎

We state this theorem only to point out that if Γ possesses sufficient regularity for length to be defined then $\mathcal{H}^1(\Gamma)$ is equal to the total length of Γ.

Theorem 3 *If Γ is a continuum with $\mathcal{H}^1(\Gamma) < \infty$, then Γ consists of a countable union of rectifiable curves together with a set of $\mathcal{H}^1$-measure zero.*

Proof: See [2], Theorem 3.14. ∎

Theorem 4 *If $\{\Gamma_n\}$ is a sequence of continuua in $\Re^n$ that converges (in Hausdorff metric) to a compact set Γ, then Γ is a continuum and $\mathcal{H}^1(\Gamma) \leq \liminf_{n\to\infty} \mathcal{H}^1(\Gamma_n)$.*

Proof: See [2], Theorem 3.18. ∎

In previous work [4,9] this result has been extended to sequences of compact sets each having uniformly bounded number of connected components. A similar result is stated here in lemma 1.

2.4 Essential Boundaries

For the treatment of boundaries of sets we will use the following notions. The *essential boundary* [3] of a set is those points where the set has density other than zero or one. To be more precise, for a measurable set $A \subset \Omega$ we set,

$$A_t = \{x \in \Omega : \lim_{\rho\to 0+} \frac{|B \cap B_\rho(x)|}{2\pi\rho^2} = t\} \quad (t \in [0,1]).$$

A_t is the set where A has density t. Federer [3] defined the essential boundary $\partial^* A$ as

$$\partial^* A = \Omega \backslash (A_0 \cup A_1)$$

The essential boundary possesses the following property,

$$\mathcal{H}^1(\partial^* A \backslash A_{\frac{1}{2}}) = 0. \tag{1}$$

Also, the set $\partial^* A$ is countably rectifiable in the sense of Federer ([3], chapter 3),

$$\partial^* A \subset \bigcup_{n=1}^{\infty} \Gamma_n \cup N$$

where the Γ_n are C^1 hypersurfaces and $\mathcal{H}^1(N) = 0$. A measurable set $A \subset \Omega$ is a *Cacciopoli set* if $\mathcal{H}^1(\partial^* A) < \infty$.

A result which characterizes the essential boundary very nicely is the following. For bounded measurable sets A, if $\mathcal{H}^1(\partial^* A) < \infty$ then,

$$\mathcal{H}^1(\Omega \cap \partial^* A) = \inf\{\liminf_{n\to\infty} \mathcal{H}^1(\Omega \cap \partial A_n) : A_n \to A \text{ locally in measure, } A_n \text{ polyhedral}\} \tag{2}$$

2.5 An Existence Theorem

The existence of minimizers of E has been shown when Ω is an open rectangle (see [6,8,11]). The theorem can be proved for more general domains but we will not concern ourselves with this issue here. The following statement is adapted from [6].

Theorem 5 *Let Ω be an open rectangle and and let $g \in L^\infty(\Omega)$. For all closed sets $\Gamma \subset \Omega$ there exists minima with Γ composed of a finite number of C^1 curves joined only at triple points with $120°$ angles or with the boundary of Ω making $90°$ angles.*

Henceforth, by a minimizer of E we mean a Γ such as described by the theorem. One would not expect that boundaries of objects, however they might be defined would satisfy the constraints existing on the minimizers of E. However, these constraints are of a local nature and it is the purpose of this paper to show that from a more global point of view the minimizers of E may be quite reasonable.

We now state a lower semi-continuity lemma which will be of use in the analysis to follow.

Lemma 1 *Let $\{C_n\}$ be a sequence of closed subsets of Ω such that each is composed of at most $N < \infty$ (N is arbitrary) connected components of a minimizer of E for some g and some λ, then there exists a subsequence (which we denote the same way) and a closed set $C \in \overline{\Omega}$ such that $d_H(C_n, C) \to 0$ and $\mathcal{H}^1(C) \leq \liminf_{n\to\infty} \mathcal{H}^1(C_n)$.*

Proof: Because of the conditions on minimizers of E we have $\mathcal{H}^1(\overline{C_n}) = \mathcal{H}^1(C_n)$ (where the closure is taken in $\Re^2$). The number of connected components of $\overline{C_n}$ is bounded above by the number of connected components of C_n. By applying theorem 1 we first extract a convergent subsequence of $\overline{C_n}$ with the limit C. In theorem 2 of [9] (or theorem 5 of [4]) it was shown under these conditions that $\mathcal{H}^1(C) \leq \liminf_{n\to\infty} \mathcal{H}^1(\overline{C_n})$. Noting $d_H(\overline{C_n}, C) = d_H(C_n, C)$ we get the desired result. ∎

2.6 Assumptions on the Domain

The assumptions we require on the domain do not go beyond those needed for theorem 5, the existence theorem. Therefore we will for convenience assume that our domain is an open rectangle. For the results of this paper we will need the following isoperimetric inequality: there is a constant $\varsigma > 0$ such that if $A \in \Omega$ is a Cacciopoli set then,

$$\mathcal{H}^1(\gamma \cap \Omega) \geq \varsigma \min\{|A|^{\frac{1}{2}}, |\Omega\backslash A|^{\frac{1}{2}}\}.$$

We remark that it is enough that this inequality be satisfied when A is a polygon for it to hold for all Cacciopoli sets.

Suppose γ is a connected component of some minimizer Γ of E which is some positive distance from $\partial\Omega$. Let O be the connected component of $\Re^2\backslash\gamma$ containing $\Re^2\backslash\Omega$. By *the set bounded by* γ we mean $F = \Re^2\backslash O = \Omega\backslash O$. If γ is not separated from the boundary of Ω by a positive distance then $\overline{\gamma} \cap \partial\Omega$ is some finite set of points $p_1, \ldots, p_m$. Since the boundary of Ω is a Jordon curve we can assume the points are ordered along the boundary. Thus $\partial\Omega\backslash\{p_1, \ldots, p_m\}$ consists of m segments of the boundary which we denote $s_1, \ldots, s_m$. For any $i \in \{1, \ldots, m\}$ we define the set O_i as the connected component of $\Re^2\backslash\{\gamma \cup (\partial\Omega\backslash s_i)\}$ containing $\Re^2\backslash\overline{\Omega}$. Set $F_i = \Omega\backslash O_i$. We can now define F, the set bounded by γ to be an F_i of minimal area (we choose it arbitrarily if it is not unique.) If $\mathcal{H}^1(\gamma) < \varsigma\sqrt{\frac{|\Omega|}{2}}$ then we can conclude from the isoperimetric inequality that F is unique. In any case we have from the isoperimetric inequality that $\mathcal{H}^1(\gamma) \geq \varsigma|F|^{\frac{1}{2}}$.

2.7 Assumptions on the Image

To prove the main result of this paper we need to make certain assumptions on the data g. The case we are interested in is one in which the image is a corrupted version of a piecewise constant L^∞ function g_c. We will define a set which we interpret as the natural candidate for a set of boundaries in the image.

We assume that Ω can be decomposed into a countable number of disjoint Cacciopoli sets A_j and that g_c is constant on each A_j. We define the boundary $\mathcal{B}_g$ to be $\Omega \cap \bigcup_j \partial^* A_j$. We assume $\mathcal{H}^1(\mathcal{B}_g) < \infty$. Without loss of generality we may assume that if $\mathcal{H}^1(\partial^* A_j \cap \partial^* A_i) > 0$ for $i \neq j$ then $g_c(A_j) \neq g_c(A_i)$ (if this fails replace A_j and A_i with their union.) The set $\{g_c(A_j)\}$ is countable and bounded; we denote it $\{a_i\}$ and define $R_i = \bigcup_{\{j: g_c(A_j) = a_i\}} A_j$.

Assumption 1 $\mathcal{H}^1(\mathcal{B}_g) < \infty$ *and* $\mathcal{H}^1(\overline{\mathcal{B}_g} \backslash \mathcal{B}_g) = 0$.

The first part of this assumption we stated earlier, the second is a mild regularity constraint and is actually used for a only small portion of the results. Furthermore the results still hold without this assumption but with it the proofs become more elementary. Without loss of generality we assume each connected component of $\Omega \backslash \overline{\mathcal{B}_g}$ is contained in some single A_j or, slightly stronger, that $(A_j)_1 \subset A_j$.

The observed image g will be a corrupted version of g_c, allowing for some smearing of the image and additive noise. To simplify the problem of controlling the effect of the smearing we make the following assumption;

Assumption 2 *There is a constant* $c_b < \infty$ *such that* $|[\mathcal{B}_g]_r \cap \Omega| < c_b r$.

This assumption can be dropped if we do not need to allow for smearing. Alternatively the decay in r can be weakened. The main reason for allowing for smearing is not to require the image to have actual jumps. The assumption is automatically satisfied for a large class of sets containing all closed sets having finite $\mathcal{H}^1$ measure and finitely many connected components. This is a consequence of the following result from the theory of Minkowski content, which we state without proof,

Proposition 1 *[3] Let Γ be a continuum in $\Re^2$ with $\mathcal{H}^1(\Gamma) < \infty$ then*

$$\lim_{\epsilon \to 0+} \frac{|[\Gamma]_\epsilon|}{2\epsilon} = \mathcal{H}^1(\Gamma)$$

Let S_r be the class of maps taking $L^\infty(\Omega)$ to $L^\infty(\Omega)$ having the property that the value of the image function at a point $x \in \Omega$ lies within the range of essential values that the argument function takes in a ball of radius r around x. This models smearing of the image and hence distortion of the boundaries. More formally $\Phi \in S_r$ iff Φ has the property $\Phi(g)(x) \in [\text{ess inf}\,\{g(x) : x \in B_r\},\ \text{ess sup}\,\{g(x) : x \in B_r\}]$. An example of such a Φ would be a smoothing operator defined using a mollifier with support lying inside the ball of radius r, but nonlinear perturbations are also allowed. Rather than prove results for a single image our results will hold for all images belonging to some class which we will now define. We assume that any image g has a representation of the form,

$$g = \Phi(g_c) + \vartheta w \tag{3}$$

for some $\Phi \in S_r$ and $w \in L^\infty$ with $\|w\|_\infty \leq 1$ and ϑ a real scalar. We will be allowing λ to tend to zero and we will need to make similar assumptions on r and ϑ. We assume therefore that there are functions $h_r : (0, \infty) \to [0, \infty)$ and $h_\vartheta : (0, \infty) \to [0, \infty)$ and positive constants $c_r, c_\vartheta < \infty$ satisfying

$$\begin{aligned} h_r(\lambda) &\leq c_r \lambda \\ h_\vartheta(\lambda) &\leq c_\vartheta \sqrt{\lambda} \\ \lim_{\lambda \to 0+} \frac{h_r(\lambda)}{\lambda} &= 0 \\ \lim_{\lambda \to 0+} \frac{h_\vartheta(\lambda)}{\sqrt{\lambda}} &= 0 \end{aligned}$$

We say $g \in \Upsilon(\lambda)$ if and only if g satisfies 3 for some Φ, w and ϑ as specified, with $r \leq h_r(\lambda)$ and $\vartheta \leq h_\vartheta(\lambda)$. For convenience we assume that $\vartheta \leq 1$ and define $K = 2 + \text{ess sup}\, g_c - \text{ess inf}\, g_c$. ($K$ bounds the gap between the maximum and minimum of g.)

In this paper we prove that in the limit $\lambda \to 0$, Γ_λ converges to $\overline{\mathcal{B}_g}$ in the topology induced by the Hausdorff metric. Furthermore the length of the optimal boundary converges to $\mathcal{H}^1(\mathcal{B}_g)$ and $\frac{1}{\sqrt{\lambda}}(f - g_c)$ converges to 0 in $L^2(\Omega)$.

3 Preliminary Bounds

In this section we prove some inequalities which will be of importance for the development of the main results, which occurs in the next section.

Theorem 6 *Given a countable set* $\{a_i : i = 0, 1, \ldots\} \subset \Re$, *a nonnegative* l_1 *sequence* $\{r_i : i = 1, 2, \ldots\}$ *and constants* $c_1, c_2 > 0$, *there exists a nondecreasing function* $h : (0, \infty) \to [0, \infty)$ *satisfying* $\lim_{t \to 0^+} \frac{h(t)}{\sqrt{t}} = 0$ *such that for any sequence* $\{p_i : i = 0, 1, \ldots\}$ *satisfying,*

$$p_0 \geq c_1 \text{ and } r_i \geq p_i \geq 0 \text{ for } i > 0;$$

$$\sum_{i=0}^{\infty} p_i = 1 \text{ and}$$

$$\sum_{i=0}^{\infty} p_i (a_i - \hat{a})^2 < c_2 t \text{ where } \hat{a} = \sum_{i=0}^{\infty} p_i a_i,$$

we have $|\hat{a} - a_0| < h(t)$.

Proof: We define the constant $b = \sum_{i=1}^{\infty} r_i$. We assume $b > 0$ (the result is trivial otherwise).

Define $h_1 : (0, \infty] \to [0, b]$ by,

$$h_1(t) = \sum_{i : 0 < |a_i - a_0| < t} r_i.$$

Clearly $h_1(t)$ is nondecreasing. We claim h_1 is continuous from the left and $\lim_{t \to 0^+} h_1(t) = 0$. For any $\epsilon > 0$, $\exists N < \infty$ such that $\sum_{i=N}^{\infty} r_i < \epsilon$. For $t \leq \min\{|a_i - a_0| : 0 < i < N\}$ we have $h_1(t) < \epsilon$ proving the second part of the claim. Given $t > 0$ let $\delta = \min\{t - |a_i - a_0| : 0 < i < N \text{ and } t - |a_i - a_0| > 0\}$; for $t' \in (t - \delta, t)$ we have $h_1(t) - h_1(t') < \epsilon$, proving the first part of the claim.

Define $h_2 : (0,\infty) \to [b^{-\frac{1}{2}}, \infty)$ by

$$h_2(t) = \sup\{c : h_1(x\sqrt{t}) \leq \frac{1}{x^2},\ \forall x < c\}$$

Since h_1 is nondecreasing and bounded above by b, h_2 is nonincreasing and bounded below by $b^{-\frac{1}{2}}$. h_2 is finite for finite t since $\lim_{x\to\infty} h_1(x) = b$. For any $N < \infty$, $\exists \eta > 0$ such that $\epsilon \leq \eta \Rightarrow h_1(\epsilon) \leq \frac{1}{N^2}$. Thus $N\sqrt{t} \leq \eta \Rightarrow h_1(N\sqrt{t}) \leq \frac{1}{N^2}$ and since h_1 is nondecreasing while $\frac{1}{x^2}$ is decreasing we have $t \leq (\frac{\eta}{N})^2 \Rightarrow h_2(t) \geq N$. We conclude $\lim_{t\to 0+} h_2(t) = \infty$. Also since h_1 is continuous from the left we have $0 < x \leq h_2(t) \Rightarrow h_1(x\sqrt{t}) \leq \frac{1}{h_2^2(t)}$.

Define $h_3 : (0,\infty) \to [b^{-\frac{1}{2}}, \infty)$, by

$$h_3(t) = \max(b^{-\frac{1}{2}}, h_2(t) - \sqrt{\frac{c_2}{c_1}})$$

Consider the case $\hat{a} \geq a_0$. We have

$$\begin{aligned}
\sum_{\{i:a_i\leq\hat{a}\}} p_i(\hat{a} - a_i) &= \sum_{\{i:0<a_i-\hat{a}<h_3(t)\sqrt{t}\}} p_i(a_i - \hat{a}) + \sum_{\{i:h_3(t)\sqrt{t}\leq a_i-\hat{a}\}} p_i(a_i - \hat{a}) \\
&\leq \sum_{\{i:0<|a_i-a_0|<h_2(t)\sqrt{t}\}} p_i|a_i - \hat{a}| + \frac{1}{h_3(t)\sqrt{t}} \sum_{i=0}^{\infty} p_i(a_i - \hat{a})^2 \\
&\leq h_1(h_2(t)\sqrt{t})h_2(t)\sqrt{t} + \frac{c_2 t}{h_3(t)\sqrt{t}} \\
&\leq (\frac{1}{h_2(t)} + \frac{c_2}{h_3(t)})\sqrt{t} \qquad (4)
\end{aligned}$$

For the first inequality we used the obvious bound; $|\hat{a} - a_0| < \sqrt{\frac{c_2}{c_1}t}$. Define $h(t) = \frac{1}{c_1}(\frac{1}{h_2(t)} + \frac{c_2}{h_3(t)})\sqrt{t}$. That h is nondecreasing follows from the fact that h_2 (and hence h_3) is nonincreasing. Also since $\lim_{t\to 0+} h_2(t) = \infty$, and hence $\lim_{t\to 0+} h_3(t) = \infty$, it follows $\lim_{t\to 0+} \frac{h(t)}{\sqrt{t}} = 0$. Now, $p_0(\hat{a} - a_0) \leq \sum_{\{i:a_i\leq\hat{a}\}} p_i(\hat{a} - a_i)$ and from 4 we get $|a_0 - \hat{a}| \leq h(t)$. The case $\hat{a} \leq a_0$ can be treated similarly yielding the same result. This completes the proof of the theorem. ∎

The following proposition provides an upper bound on the minimum energy attainable for any $g \in \Upsilon(\lambda)$.

Proposition 2 *There is a constant $c_E < \infty$ and a function $U : (0, \infty) \to [0, \infty)$ satisfying $U(\lambda) \le c_E \lambda$ and $\lim_{\lambda \to 0+} \frac{U(\lambda)}{\lambda} = \mathcal{H}^1(\mathcal{B}_g)$ such that $\sup_{g \in \Upsilon(\lambda)} \{E^*(g, \lambda)\} \le U(\lambda)$.*

Proof: For any measurable set $A \subset \Omega$ we have the following,

$$\begin{aligned}\int_A (g - g_c)^2 &= \int_A \chi_{[\mathcal{B}_g]_r} (g - g_c)^2 + \int_A (1 - \chi_{[\mathcal{B}_g]_r})(\vartheta w)^2 \\ &\le K^2 c_b r + \vartheta^2 |A|\end{aligned}$$

where we invoke assumption 2. We now have

$$E^*(g, \lambda) \le E(g_c, \overline{\mathcal{B}_g}) \le K^2 c_b h_r(\lambda) + h_\vartheta^2(\lambda)|\Omega| + \lambda \mathcal{H}^1(\mathcal{B}_g)$$

The result now follows from our assumptions on h_r and h_ϑ. ∎

In the next proposition we get a bound on how closely f tracks g_c in regions where a portion the segmentation overlaps significantly with a region in which g_c is constant.

Proposition 3 *Given $\xi > 0$ and $i \ge 0$ there exists a function $H : (0, \infty) \to [0, \infty)$ satisfying $\lim_{\lambda \to 0+} \frac{H(\lambda)}{\sqrt{\lambda}} = 0$ such that if Γ_λ is a minimizer of E for some $g \in \Upsilon(\lambda)$ and Ω_k is a connected component of $\Omega \backslash \Gamma_\lambda$ satisfying $|\Omega_k \cup R_i| \ge \xi$, then*

$$|f(\Omega_k) - a_i| \le H(\lambda)$$

Proof: For convenience of notation we set $i = 0$ and re-enumerate the other a_i starting from 1. For each Ω_k in a segmentation we define the constants, $p_i^k = \frac{|\Omega_k \cap R_i|}{|\Omega_k|}$. Note that $\sum_{i=1}^{\infty} p_i^k = 1$.

Let $\hat{a} = \sum_{i=0}^{\infty} p_i^k a_i = \frac{1}{|\Omega_k|} \int_{\Omega_k} g_c$. We write,

$$|f(\Omega_k) - a_0| \le |f(\Omega_k) - \hat{a}| + |\hat{a} - a_0|.$$

We can bound the first term as follows,

$$\begin{aligned}|f(\Omega_k) - \hat{a}| &= \left|\frac{\vartheta}{|\Omega_k|} \int_{\Omega_k} w + \frac{1}{|\Omega_k|} \int_{\Omega_k} (\Phi_r g_c - g_c)\right| \\ &\le h_\vartheta(\lambda) + \frac{K c_b}{\xi} h_r(\lambda).\end{aligned}$$

To bound the second term we proceed

$$\begin{aligned}
\sum_{i=0}^{\infty} p_i^k(\hat{a} - a_i)^2 &= \int_{\Omega_k} (\hat{a} - g_c)^2 \\
\frac{1}{2}\int_{\Omega_k} (\hat{a} - g_c)^2 &\leq \int_{\Omega_k} (f - g)^2 + \int_{\Omega_k} (g - g_c)^2 \\
&\leq U(\lambda) + K^2 c_b h_r(\lambda) + |\Omega_k| h_\vartheta^2(\lambda)
\end{aligned}$$

where U is the function from proposition 2. Applying our assumptions and proposition 2 we see that there is a constant $c > 0$ such that

$$\sum_{i=0}^{\infty} p_i^k(\hat{a} - a_i)^2 \leq c\lambda$$

Define $r_i = \min\{1, \frac{|R_i|}{\xi}\}$. Clearly $p_i^k \leq r_i$ and we have $\sum_{i=1}^{\infty} r_i < \infty$. Note also that $p_0^k \geq \frac{\xi}{|\Omega|}$. We can now apply theorem 6 to conclude there exists a function $h : (0, \infty) \to [0, \infty)$ satisfying $\lim_{\lambda \to 0+} \frac{h(\lambda)}{\sqrt{\lambda}} = 0$ such that $|\hat{a} - a_0| \leq h(\lambda)$. Set $H = h + h_\vartheta + \frac{K c_b}{\xi} h_r$ and the result follows. ∎

Corollary *Under the conditions of proposition 3 we have*

$$\int_{\Omega_k} (a_i - g)^2 - \int_{\Omega_k} (f - g)^2 \leq |\Omega_k| H^2(\lambda)$$

Proof:

$$(f - g)^2 - (a_i - g)^2 = (a_i - f)^2 + 2(a_i - f)(f - g)$$

Since $a_i - f$ is constant in Ω_k and $f(\Omega_k) = \frac{1}{|\Omega_k|}\int_{\Omega_k} g$ we get,

$$\int_{\Omega_k} [(f - g)^2 - (a_i - g)^2] = \int_{\Omega_k} (a_i - f)^2.$$

Now apply proposition 3. ∎

4 Main Results

The goal of this section is to prove the following theorem,

Theorem 7 *Under our stated assumptions, as $\lambda \to 0$ $\{\Gamma_\lambda\}$ converges to $\overline{\mathcal{B}_g}$ with respect to the Hausdorff metric, and $\mathcal{H}^1(\Gamma_\lambda) \to \mathcal{H}^1(\mathcal{B}_g)$. We mean by this that for any $\epsilon > 0$ there exists $\eta > 0$ such that if $\lambda < \eta$ and Γ_λ is a minimizer of E for some $g \in \Upsilon(\lambda)$, then $d_H(\Gamma_\lambda, \mathcal{B}_g) < \epsilon$ and $|\mathcal{H}^1(\Gamma_\lambda) - \mathcal{H}^1(\mathcal{B}_g)| < \epsilon$.*

The proof of this theorem is quite long so we have broken it up into several sub-theorems. We assume throughout that $r \leq h_r(\lambda)$ and $\vartheta \leq h_\vartheta(\lambda)$.

The first lemma of this section shows that except for vanishingly small pieces of the boundary Γ_λ, Γ_λ lies in some neighborhood of the "true" boundaries $\mathcal{B}_g$.

Lemma 2 *For any $\epsilon, \gamma > 0$, there exists a constant $\eta > 0$ such that if $\lambda < \eta$ and Γ_λ is a minimizer of E for some $g \in \Upsilon(\lambda)$, then $\tilde{\Gamma}_\lambda \subset [\mathcal{B}_g]_\epsilon$ where $\tilde{\Gamma}_\lambda$ is the union of all the connected components of Γ_λ having $\mathcal{H}^1$ measure greater than γ.*

Proof: Without loss of generality we assume $\gamma \leq \frac{\epsilon}{4}$. Assuming the lemma is false we can find a sequence $\tilde{\Gamma}_{\lambda_n}$ with $\lambda_n \downarrow 0$ such that $\tilde{\Gamma}_{\lambda_n} \not\subset [\mathcal{B}_g]_\epsilon$ for each n. In general $\mathcal{H}^1(\Gamma_\lambda) \leq E^*(g,\lambda)/\lambda \leq c_E$, by proposition 2. Thus the number of connected components of $\tilde{\Gamma}_{\lambda_n}$ is bounded above by $\frac{c_E}{\gamma}$. Applying lemma 1 we can assume that the $\tilde{\Gamma}_{\lambda_n}$ converge with respect to the Hausdorff metric to a closed set $\tilde{\Gamma} \subset \overline{\Omega}$ satisfying $\mathcal{H}^1(\tilde{\Gamma}) < \infty$. It follows that there is an $x \in \tilde{\Gamma}$ such that $\operatorname{dist}(x, \mathcal{B}_g) \geq \epsilon$. By translation we can assume $x = 0$ and we henceforth drop it from the notation. It is clear that $\liminf_{n\to\infty} \mathcal{H}^1(\tilde{\Gamma}_{\lambda_n} \cap B_\gamma) \geq \gamma$ and hence

$$\liminf_{n\to\infty} \mathcal{H}^1(\Gamma_{\lambda_n} \cap B_{\epsilon/4}) \geq \gamma \tag{5}$$

We can find δ, $0 < \delta < \epsilon/2$ sufficiently small so that

$$|\overline{[\tilde{\Gamma}]_\delta} \cap B_{\epsilon/2}| < \gamma\frac{\epsilon}{2^5} \tag{6}$$

We know $0 = x \in \overline{A_j}$ for some A_j. $A_j \backslash \tilde{\Gamma}$ is the union countable number of disjoint connected open sets. Only finitely many of these sets are not contained in $[\tilde{\Gamma}]_\delta$. Let m be the number of these sets that have nonempty intersection with $B_{\epsilon/2}$. We can find points

$y_1, \dots y_m$ such that y_i lies in the interior of the ith component and $\min_{i \in 1,\dots,M} \operatorname{dist}(y_i, \tilde{\Gamma}) \geq \delta$. For n sufficiently large $\tilde{\Gamma}_{\lambda_n} \subset [\tilde{\Gamma}]_\delta$ and we can then define $\tilde{\Omega}_i^n$ to be the connected component of $\Omega \backslash \tilde{\Gamma}_{\lambda_n}$ which contains y_i. (They may be identical for different i .) Each $\tilde{\Omega}_i^n$ may contain connected components of Γ_{λ_n} having $\mathcal{H}^1$ measure less than or equal to γ. Let F_n be the union of the sets bounded by these components bound (see the section entitled Assumptions on the Domain for an explanation of this terminology). It follows that $\tilde{\Omega}_i^n \backslash F_n$ is some connected component of the segmentation i.e. some $\Omega_k(n)$, which we now denote $\Omega_{k_i}^n$. By reordering the $\Omega_{k_i}^n$ with respect to i and choosing an appropriate subsequence (which we still index by n) we can find $m' \leq m$ and $\xi > 0$ such that;

$$\lim_{n\to\infty} |\Omega_{k_i}^n \cup R_l| = 0 \quad \text{for} \quad i \in 1, \dots, m'$$

$$\liminf_{n\to\infty} |\Omega_{k_i}^n \cup R_l| \geq \xi \quad \text{for} \quad i \in m'+1, \dots, m$$

where l is given by $g_c(A_j) = a_l$, (and $A_j \in R_l$). Define S_n and T_n by,

$$S_n = \cup_{i=1}^{m'} \overline{\Omega_{k_i}^n}$$

$$T_n = \cup_{i=m'+1}^{m} \Omega_{k_i}^n$$

Note that $|\overline{\Omega_{k_i}^n}| = |\Omega_{k_i}^n|$ so

$$\lim_{n\to\infty} |S_n \cap R_l| = 0 \tag{7}$$

Let H be the function of proposition 3 with ξ as above and $i = l$. There exists N such that if $n > N$ then the following are all satisfied;

$$\begin{aligned}
|(B_{\epsilon/2} \backslash B_{\epsilon/4}) \cap ([\tilde{\Gamma}]_\delta \cup S_n)| &< \gamma \frac{\epsilon}{2^4} && (8)\\
\mathcal{H}^1(\Gamma_{\lambda_n} \cap B_{\epsilon/4}) &> \frac{3}{4}\gamma && (9)\\
\tilde{\Gamma}_{\lambda_n} &\subset [\tilde{\Gamma}]_\delta && (10)\\
\int_{T_n} (a_i - g)^2 - (f_n - g)^2 &\leq |T_n| H^2(\lambda) && (11)\\
h_r(\lambda_n) &< \frac{\epsilon}{4} && (12)
\end{aligned}$$

8 follows from 6 and 7, 9 follows from 5 and 10 follows by definition of $\tilde{\Gamma}$. 11 follows from the corollary to proposition 3.

Consider any such $n > N$. Since $\mathcal{H}^1(\Gamma_{\lambda_n} \cap \partial B_r) > 0$ for at most countably many $\rho \in [\epsilon/4, \epsilon/2]$ we can conclude from 8 that for some $\rho_n \in [\epsilon/4, \epsilon/2]$

$$\mathcal{H}^1(\partial B_{\rho_n} \cap (\overline{[\tilde{\Gamma}]_\delta} \cup S_n \cup \Gamma_{\lambda_n})) < \frac{\gamma}{4} \tag{13}$$

Note by 10 that any connected component of Γ_{λ_n} which has nonempty intersection with $\partial B_{\rho_n} \backslash [\tilde{\Gamma}]_\delta$ has $\mathcal{H}^1$ measure and hence diameter less than or equal to $\gamma(\leq \epsilon/4)$, and since $\rho_n \leq \epsilon/2$ it must lie entirely within $\Omega \cap \overline{B_{\frac{3}{4}\epsilon}}$ which in turn lies in A_j. Let $W_n = \cup\{\Omega_k(n) : \Omega_k(n) \subset B_{\frac{3}{4}\epsilon}\}$. We have

$$\overline{B_{\rho_n}} \subset T_n \cup S_n \cup W_n \cup \Gamma_{\lambda_n} \cup \overline{[\tilde{\Gamma}]_\delta}$$

Now, define

$$\Gamma'_{\lambda_n} = (\Gamma_{\lambda_n} \backslash B_{\rho_n}) \cup (\partial B_{\rho_n} \cap (\overline{[\tilde{\Gamma}]_\delta} \cup S_n \cup \Gamma_{\lambda_n}))$$

Let

$$f'_n(x) = \begin{cases} a_i & x \in T_n \cup W_n \cup B_{\rho_n} \\ f_n & \text{elsewhere} \end{cases}$$

It follows that f'_n is constant on each connected component of $\Omega \backslash \Gamma'_{\lambda_n}$.

From 13 and 9 it follows

$$\mathcal{H}^1(\Gamma_{\lambda_n}) - \mathcal{H}^1(\Gamma') \geq \frac{\gamma}{2}$$

and from 11 and 12 and the fact $\Omega \cap B_{\frac{3}{4}\epsilon} \subset R_l$ we get

$$\begin{aligned} \int_\Omega (f'_n - g)^2 - \int_\Omega (f_n - g)^2 &= \int_{T_n \cup W_n \cup B_{r_n}} (f'_n - g)^2 - (f_n - g)^2 \\ &\leq \int_{T_n} (a_i - g)^2 - (f_n - g)^2 + \int_{B_{\frac{3}{4}\epsilon} \backslash T_n} (a_i - g)^2 \\ &\leq |T_n| H^2(\lambda_n) + \pi(\frac{3}{4}\epsilon)^2 h_\vartheta^2(\lambda_n). \end{aligned}$$

We can now write,

$$E(f'_n, \Gamma'_n) - E(\Gamma_{\lambda_n}) \leq -\lambda_n \frac{\gamma}{2} + |T_n| H^2(\lambda_n) + \pi(\frac{3}{4}\epsilon)^2 h_\vartheta^2(\lambda_n)$$

which is negative for n sufficiently large (and λ_n sufficiently small) contradicting the optimality of Γ_{λ_n}. We conclude $\tilde{\Gamma}_{\lambda_n} \cap \Omega \backslash [\mathcal{B}_g]_\epsilon = \emptyset$ for all n sufficiently large. ∎

In the following theorem we further show that there are no vanishingly small pieces of the boundary remaining outside some neighborhood of $\mathcal{B}_g$ when λ is small enough.

Theorem 8 *For any $\epsilon > 0$ there exists a constant $\eta > 0$ such that if $\lambda < \eta$ then $\Gamma_\lambda \subset [\mathcal{B}_g]_\epsilon$.*

Proof: Assume the theorem is false. Only finitely many A_j satisfy $A_j \not\subset [\mathcal{B}_g]_\epsilon$. Thus there exists some such A_j and a sequence Γ_{λ_n} with $\lambda_n \downarrow 0$ such that $\Gamma_{\lambda_n} \cap A_j \backslash [\mathcal{B}_g]_\epsilon \neq \emptyset$ for each n. Let $\{C_i^n\}$ be the set of connected components of Γ_{λ_n} satisfying $C_i^n \cap A_j \backslash [\mathcal{B}_g]_{\epsilon/2} \neq \emptyset$ and let $\hat{\Gamma}_{\lambda_n} = \Gamma_{\lambda_n} \backslash \cup_i C_i^n$. From lemma 2 we conclude

$$\lim_{n\to\infty} \max_i (\mathcal{H}^1(C_i^n)) = 0. \tag{14}$$

Let $2\xi = |A_j \backslash [\mathcal{B}_g]_{\epsilon/2}|$ (> 0). By lemma 2 for n sufficiently large there is a is a connected component of $\Omega \backslash \hat{\Gamma}_{\lambda_n}$, $\hat{\Omega}_k^n$ which satisfies $|\hat{\Omega}_k^n \cap A_j| > 2\xi$. Some subset of the Ω_k^n lying in $\hat{\Omega}_k^n$, whose union we denote by O_n, are the sets bounded by the C_i^n. It follows from the isoperimetric inequality that $|O_n| \leq \frac{1}{\varsigma^2}(\max_i \mathcal{H}^1(C_i^n)) \sum_i \mathcal{H}^1(C_i^n)$. Since the sum is bounded we have by 14 that for n large enough $|O_n| \leq \xi$. Hence there is some $\Omega_k^n \subset \hat{\Omega}_k^n$ satisfying $|\Omega_k^n \cap A_j| \geq \xi$. Let H be the function from proposition 3 with ξ as above and i defined by $A_j \subset R_i$. Assuming n is large enough so that $h_r(\lambda_n) < \epsilon - \max_i \mathcal{H}^1(C_i^n)$ we have,

$$\begin{aligned} E(\hat{\Gamma}_{\lambda_n}) - E(\Gamma) &\leq |O_n|(|f_n(\Omega_k) - a_i|^2 + \vartheta^2) - \lambda_n \sum_i \mathcal{H}^1(C_i^n) \\ &\leq [\frac{1}{\varsigma^2} \max_i \mathcal{H}^1(C_i^n)(H^2(\lambda_n) - h_\vartheta^2(\lambda_n)) - \lambda_n] \sum_i \mathcal{H}^1(C_i^n) \end{aligned}$$

Since the term in square brackets is negative for n sufficiently large while the sum is positive we get a contradiction of the optimality of Γ. This completes the proof of the theorem. ∎

The next lemma shows that every point in $\mathcal{B}_g$ must have a point in Γ_λ close to it when λ is small.

Lemma 3 *For all $\epsilon > 0$, $\exists \eta > 0$ such that if $\lambda < \eta$ then $\mathcal{B}_g \subset [\Gamma_\lambda]_\epsilon$.*

Proof: Suppose the lemma is false. There then exists a sequence of Γ_{λ_n}, $\lambda_n \downarrow 0$, an $x \in \mathcal{B}_g$ and a $\rho > 0$ such that $B_\rho(x) \cap \Gamma_{\lambda_n} = \emptyset$ for all n. We can find at least two values i_1, i_2 such that for some $\xi > 0$,

$$\min_{l=1,2} |B_\rho(x) \cap R_{i_l}| = \xi > 0$$

Let $\delta a = |a_{i_1} - a_{i_2}|$. Clearly f_n is constant in $B_\rho(x)$ and for at least one of the i_l we have $f(x) - a_{i_l} \geq \frac{\delta a}{2}$. But from this we conclude

$$E(\Gamma_{\lambda_n}) \geq \int_{B_\rho(x)} (g_c - f_n)^2 - \int_{B_\rho(x)} (g - g_c)^2 \geq \xi(\frac{\delta a}{2})^2 - Kc_b h_r(\lambda_n) - |B_\rho(x)| h_\vartheta^2(\lambda_n)$$

which contradicts the bound $E(\Gamma_{\lambda_n}) \leq U(\lambda_n) \leq c_E \lambda_n$ given in proposition 2, when λ_n is sufficiently small. ∎

The last two lemmas establish that the length of Γ_λ converges to $\mathcal{H}^1(\mathcal{B}_g)$ as λ tends to zero.

Lemma 4 *For any $\epsilon > 0$, $\exists \eta > 0$ such that if $\lambda < \eta$ then*

$$\mathcal{H}^1(\Gamma_\lambda) > \mathcal{H}^1(\mathcal{B}_g) - \epsilon.$$

Proof: The symmetric difference between $\overline{\mathcal{B}_g}$ and $\cup_{i \neq j} \partial^* A_i \cap \partial^* A_j$ is an $\mathcal{H}^1$ negligible because of our assumptions and the property of essential boundaries stated in equation 1. Each $\overline{\partial^* A_i \cup \partial^* A_j}$ can be written as a countable union of rectifiable curves meeting only at their end points together with a $\mathcal{H}^1$ negligible set by theorem 3. Thus in general we can write $\overline{\mathcal{B}_g} = N \cup \bigcup_{i=1}^\infty E_i$ where N has negligible $\mathcal{H}^1$ measure and the E_i's are rectifiable curves joined only at their end points such that for each we can find A_{j_1}, A_{j_2} such that $\delta a_i = |g(A_{j_1}) - g(A_{j_2})| > 0$ and $E_i \subset \overline{\partial^* A_{j_1} \cap \partial^* A_{j_2}}$. Define $h(E_i) = \min(|A_{j_1}|, |A_{j_2}|)$.

Suppose the lemma is false. Then there exists a sequence of minimizers Γ_{λ_n} such that $\mathcal{H}^1(\Gamma_{\lambda_n}) \leq \mathcal{H}^1(\mathcal{B}_g) - \epsilon$. We can find an $M < \infty$ so that $\sum_{i=1}^M \mathcal{H}^1(E_i) > \mathcal{H}^1(\mathcal{B}_g) - \epsilon/2$. Let $2\xi = \min_{i \in 1,\ldots,M} h(E_i)$ and let $\tilde{\Gamma}_{\lambda_n} = \Gamma_{\lambda_n} \setminus \cup_i C_i^n$ where the C_i^n are the connected components of Γ_{λ_n} satisfying $\mathcal{H}^1(C_i^n) \leq \xi \varsigma^2 / 2c_E$ where c_E is the constant from proposition 2. By theorem 1 we can find a subsequence of the $\tilde{\Gamma}_{\lambda_n}$ (still indexed by n) which converges in the Hausdorff metric to some $\tilde{\Gamma}_l$; we claim $\cup_{i=1}^M E_i \subset \tilde{\Gamma}_l$. Assume this were not the case. $\exists x \in E_i$ for some $i \le M$ such that $\mathrm{dist}(x, \tilde{\Gamma}_l) > 0$. We can find compact connected sets K_1, K_2 such that $K_1 \subset A_{j_1}$, $K_2 \subset A_{j_2}$ and $|K_1|, |K_2| > \xi$. From the isoperimetric inequality we conclude that the total area of the sets bounded by the C_i^n is less than or equal to $\frac{1}{\varsigma^2}(\max_i \mathcal{H}^1(C_i^n)) \sum_i \mathcal{H}^1(C_i^n) \le \xi/2$, since the sum is bounded by c_E. It follows that for n sufficiently large there is a single $\Omega_k(n)$ such that

$$|\Omega_k(n) \cap K_i| \ge \frac{\xi}{2},\ i = 1, 2$$

and hence

$$E(\Gamma_{\lambda_n}) \ge \int_{\Omega_k(n)} (g - f)^2 \ge \frac{\xi}{4}(\frac{\delta a_i}{2})^2 - K c_b h_r(\lambda_n) - h_\vartheta^2(\lambda_n)|\Omega_k(n)|$$

which, for n large enough, contradicts the optimality of Γ_{λ_n}. Thus $\cup_{i=1}^M E_i \subset \tilde{\Gamma}_l$ and by lemma 1 we have along the subsequence,

$$\liminf_{n \to \infty} \mathcal{H}^1(\tilde{\Gamma}_{\lambda_n}) \ge \mathcal{H}^1(\mathcal{B}_g) - \epsilon/2$$

and hence

$$\liminf_{n \to \infty} \mathcal{H}^1(\Gamma_{\lambda_n}) \ge \mathcal{H}^1(\mathcal{B}_g) - \epsilon/2$$

which gives us a contradiction. ∎

Lemma 5 *For any $\epsilon > 0$, $\exists \eta > 0$ such that if $\lambda < \eta$ then*

$$\mathcal{H}^1(\Gamma_\lambda) < \mathcal{H}^1(\mathcal{B}_g) + \epsilon$$

Proof: This lemma is a simple consequence of proposition 2 which yields $\mathcal{H}^1(\Gamma_\lambda) \le \frac{E^*(g,\lambda)}{\lambda} \le \frac{U(\lambda)}{\lambda} \le \mathcal{H}^1(\mathcal{B}_g) + \epsilon$ for all λ sufficiently small. ∎

We now have available all the facts needed to fulfill the stated goal of this section.

Proof of Theorem 7: Theorem 8 and lemma 3 establish $d_H(\Gamma_\lambda, \overline{\mathcal{B}_g}) < \epsilon$ while lemmas 4 and 5 prove $|\mathcal{H}^1(\Gamma_\lambda) - \mathcal{H}^1(\mathcal{B}_g)| < \epsilon$ for all $\lambda < \eta$ for some $\eta > 0$. ∎

We remark that in the course of the proof we have shown that $|E^*(g,\lambda) - \mathcal{H}^1(\mathcal{B}_g)| < \epsilon$ as well as $|\mathcal{H}^1(\Gamma_\lambda) - \mathcal{H}^1(\mathcal{B}_g)| < \epsilon$; we conlude from this that $\frac{1}{\lambda}\int_\Omega (f-g)^2 < 2\epsilon$. Since $\int_\Omega (g-g_c)^2 \leq U(\lambda) < \epsilon$ (see proposition 2) we also have $\frac{1}{\lambda}\int_\Omega (f-g_c)^2 < 6\epsilon$.

5 Conclusions

In this paper we have shown that the variational "cartoon" model is asymptotically faithful in the sense that images which are approximately "cartoon-like" yield solutions close to image, the boundaries closely matching the discontinuity set. We plan in future work to extend this work to remove the asymptotic nature of the result. It is known, for example, that if the image is of a circle, e.g. $\Omega = \{\|x\|_1 < 1\}$ and $g = 1$ for $\|x\|_2 < \frac{1}{2}$ and 0 elsewhere, then the only possible minimizers of E_0 are $\Gamma = \emptyset$ and $\Gamma = \{\|x\|_2 = \frac{1}{2}\}$. For large λ the unique minimizer is $\Gamma = \emptyset$ while for small λ the unique minimizer is $\Gamma = \{\|x\|_2 = \frac{1}{2}\}$. There is some critical value of λ where both solutions are minimizers. We conjecture that if g is piecewise constant and the set $\mathcal{B}_g$ is composed of a finite number of C^2 curves having uniformly bounded curvature with endpoints forming singularities only such as those possessed by minimizers of E_0, then there is some finite λ^* such that if $\lambda < \lambda^*$ then the unique minimizer of E_0 is $\Gamma = \mathcal{B}_g$.

Extensions to the piecewise smooth case (where the functional to be minimized includes a term weighting the gradient of f) are forthcoming. This work may be of aid in deciding values for the parameters of the functional, designing algorithms, and in understanding and predicting the behavior of the variational approach.

References

[1] Blake, A and Zisserman, A. *Visual Reconstruction*
MIT Press, Cambridge, 1987

[2] Falconer, K.J. *The Geometry of Fractal Sets*
Cambridge University Press, 1985

[3] Federer, H. *Geometric Measure Theory*
Springer-Verlag, 1969

[4] Kulkarni, S. R., Mitter, S. K., Richardson, T. J., An Existence Result and Lattice Approximations for a Variational Problem Arising in Computer Vision,
Proceedings of the Signal Processing Workshop, Inst. for Mathematics and Applications, U. of Minnesota, 1989. Also, Center for Intelligent Control Systems Report CICS-P-123. MIT. 1989.

[5] Marr, D. *Vision*
W.H. Freeman an Co. 1982

[6] Morel, J-M., and Solimini, S. Segmentation of Images by Variational Methods: a Constructive Approach.
Revista Matematica de la Universidad Complutense de Madrid, Vol. 1, num 1,2,3; 1988.

[7] Mumford, D. and Shah, J. Boundary detection by minimizing functionals,
IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco 1985.

[8] Mumford, D. and Shah, J. Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems.
Submitted to *Communications on Pure and Applied Mathematics*. Also, Center for Intelligent Control Systems Report CICS-P-68. MIT. 1988

[9] Richardson, T.J. Existence Result for a Problem Arising in Computer Vision
Center for Intelligent Control Systems Report CICS-P-63. MIT. 1988

[10] Rogers, C.A. *Hausdorff Measure.*
Cambridge University Press. 1970

[11] Wang, Y. Unpublished notes (Harvard University).